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Perturbation Theory

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A central problem [1] in the perturbation theory of unitary transformations is studied. The problem is formulated in the vector generalization [2] of the theory of square summable power series [3].

In the usual notation for square summable power series, $\mathscr{C}$ is a fixed Hilbert space which is used as a coefficient space. A vector is always an element of this space. An operator is a bounded linear transformation of vectors into vectors. For every vector a, $\bar{a}$ is the linear functional on vectors defined by the inner product $\bar{a}b = \langle b, a\rangle$ for every vector b. The absolute value symbol is used for the norm of a vector and for the operator norm of an operator. If a and b are vectors, $a\bar{b}$ is the operator defined by $(a\bar{b})\, c = a(\bar{b}c)$ for every vector c.

The underlying Hilbert space is the space $\mathscr{C}(z)$ of square summable power series $f(z) = \sum a_n z^n$ with vector coefficients, $\| f(z)\|^2 = \sum | a_n |^2$. Basic constructions are made using any power series $B(z)$ with operator coefficients such that $B(z) f(z)$ belongs to $\mathscr{C}(z)$ whenever $f(z)$ belongs to $\mathscr{C}(z)$ and such that the inequality $\| B(z) f(z)\| \leqslant \| f(z)\|$ holds for every element $f(z)$ of $\mathscr{C}(z)$. An equivalent condition is that $B(z)$ converges and represents a function which is bounded by one in the unit disk. The B-norm of an element $f(z)$ of $\mathscr{C}(z)$ is defined by

$$\| f(z)\|_B^2 = \sup[\| f(z) + B(z)\, g(z)\|^2 - \| g(z)\|^2]$$

where the supremum is taken over all elements $g(z)$ of $\mathscr{C}(z)$. The set $\mathscr{H}(B)$ of elements of $\mathscr{C}(z)$ with finite B-norm is a Hilbert space in the B-norm. The space is contained in $\mathscr{C}(z)$ and the inclusion does not increase norms. The space contains $[1 - B(z)\, \bar{B}(w)]\, c/(1 - z\bar{w})$ for every vector c when $| w | < 1$ and the identity

$$\bar{c} f(w) = \langle f(z), [1 - B(z)\, \bar{B}(w)]\, c/(1 - z\bar{w})\rangle_B$$

holds for every element $f(z)$ of the space.

The related space $\mathscr{L}(\varphi)$ is constructed from any power series $\varphi(z)$, having operator coefficients, which converges and represents a function with nonnegative real part in the unit disk. The elements of the space are power series with vector coefficients which converge and represent vector valued analytic functions in the

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disk. The space contains $[\varphi(z) + \bar{\varphi}(w)]\, c/(1 - z\bar{w})$ for every vector c when $|w| < 1$ and the identity

$$2\bar{c}f(w) = \langle f(z), [\varphi(z) + \bar{\varphi}(w)]\, c/(1 - z\bar{w})\rangle_{\mathscr{L}(\varphi)}$$

holds for every element $f(z)$ of the space. If $\mathscr{L}(\varphi)$ is a given space, a space $\mathscr{H}(B)$ exists such that

$$B(z) = [1 - \varphi(z)]/[1 + \varphi(z)].$$

The transformation which takes $f(z)$ into $[1 + B(z)]\, f(z)$ is an isometry of $\mathscr{L}(\varphi)$ onto $\mathscr{H}(B)$.

The space $\mathscr{L}(\varphi)$ is interesting for its relation to the difference–quotient transformation. The series $[f(z) - f(0)]/z$ belongs to the space whenever $f(z)$ belongs to the space, and the inequality

$$\| [f(z) - f(0)]/z \|_{\mathscr{L}(\varphi)} \leqslant \| f(z) \|_{\mathscr{L}(\varphi)}$$

holds for every element $f(z)$ of the space. The transformation which takes $f(z)$ into $[f(z) - f(0)]/z$ has an isometric adjoint in the space. This property essentially characterizes such spaces: A Hilbert space of formal power series with vector coefficients such that the transformation which takes $f(z)$ into $f(0)$ of the space into $\mathscr{C}$ is continuous is isometrically equal to a space $\mathscr{L}(\varphi)$ if it is invariant under the transformation which takes $f(z)$ into $[f(z) - f(0)]/z$ and if the transformation has an isometric adjoint in the space.

A space derived from a space $\mathscr{L}(\varphi)$ is needed for the study of unitary transformations because the difference–quotient transformation need not be isometric in the space. The extension space $\mathscr{E}(\varphi)$ of $\mathscr{L}(\varphi)$ is a Hilbert space whose elements are pairs $(f(z), g(z))$ of power series with vector coefficients. The pair $(f(z), g(z))$ belongs to the space if $f(z)$ belongs to $\mathscr{L}(\varphi)$ and if $g(z) = \sum a_n z^n$ where

$$z^{n+1} f(z) + a_0 z^n + \cdots + a_n$$

belongs to $\mathscr{L}(\varphi)$ for every nonnegative integer n and the sequence of norms

$$\| z^{n+1} f(z) + a_0 z^n + \cdots + a_n \|_{\mathscr{L}(\varphi)}$$

is bounded. The norm sequence is nondecreasing because the difference–quotient transformation does not increase norms. The norm in $\mathscr{E}(\varphi)$ is defined by

$$\| (f(z), g(z)) \|_{\mathscr{E}(\varphi)} = \lim \| z^{n+1} f(z) + a_0 z^n + \cdots + a_n \|_{\mathscr{L}(\varphi)} \,.$$

The transformation which takes $(f(z), g(z))$ into

$$([f(z) - f(0)]/z,\; z g(z) + f(0))$$

is unitary in $\mathscr{E}(\varphi)$. The transformation which takes $(f(z), g(z))$ into $f(z)$ is a partial isometry of $\mathscr{E}(\varphi)$ onto $\mathscr{L}(\varphi)$.

For every space $\mathscr{L}(\varphi)$, a space $\mathscr{L}(\varphi^*)$ exists, $\varphi^*(z) = \sum \bar{\varphi}_n z^n$ if $\varphi(z) = \sum \varphi_n z^n$. The pair

$$([\varphi(z) + \bar{\varphi}(w)]\, c/(1 - z\bar{w}),\ [\varphi^*(z) - \bar{\varphi}(w)]\, c/(z - \bar{w}))$$

belongs to $\mathscr{E}(\varphi)$ for every vector c when $|w| < 1$. The identity

$$2\bar{c}f(w) = \langle (f(z), g(z)), ([\varphi(z) + \bar{\varphi}(w)]\, c/(1 - z\bar{w}), [\varphi^*(z) - \bar{\varphi}(w)]\, c/(z - \bar{w}))\rangle_{\mathscr{E}(\varphi)}$$

holds for every element $(f(z), g(z))$ of the space. The transformation which takes $(f(z), g(z))$ into $(g(z), f(z))$ is an isometry of $\mathscr{E}(\varphi)$ onto $\mathscr{E}(\varphi^*)$.

In the perturbation theory of unitary transformations, pairs of spaces $\mathscr{L}(\varphi)$ and $\mathscr{L}(\psi)$ arise such that $\varphi(z)$ and $\psi(z)$ are reciprocals. The transformation which takes $(f(z), g(z))$ into $(\varphi(z) f(z), -\varphi^*(z)\, g(z))$ is then an isometry of $\mathscr{E}(\psi)$ onto $\mathscr{E}(\varphi)$. This allows the unitary transformations which take $(f(z), g(z))$ into $([f(z) - f(0)]/z,\ zg(z) + f(0))$ in the spaces $\mathscr{E}(\varphi)$ and $\mathscr{E}(\psi)$ to be regarded as perturbations of each other. A fundamental identity of perturbation theory states that

$$\begin{aligned} 2\bar{u}(0) f(0) = {} & \langle (f(z), g(z)), (\varphi(z)\, u(z),\ -\varphi^*(z)\, v(z))\rangle_{\mathscr{E}(\varphi)} \\ & - \langle ([f(z) - f(0)]/z,\ zg(z) + f(0)), (\varphi(z)\, [u(z) - u(0)]/z, \\ & - \varphi^*(z)\, [zv(z) + u(0)])\rangle_{\mathscr{E}(\varphi)} \end{aligned}$$

for all elements $(f(z), g(z))$ of $\mathscr{E}(\varphi)$ and $(u(z), v(z))$ of $\mathscr{E}(\psi)$. This situation forms a canonical model of the general perturbation problem for unitary transformations.

If U and V are unitary transformations in a Hilbert space and if no nonzero element of the space is in the kernel of $U^n - V^n$ for every integer n, then U and V are unitarily equivalent to the above unitary transformations in $\mathscr{E}(\varphi)$ and $\mathscr{E}(\psi)$ for some choice of $\varphi(z)$ and $\psi(z)$. The unitary equivalences are consistent with the above isometry of $\mathscr{E}(\varphi)$ onto $\mathscr{E}(\psi)$.

The central problem of perturbation theory is concerned with the properties of a transformation, called the wave limit, which formally gives unitary equivalence between prescribed parts of the spectra of any two given unitary transformations in a Hilbert space. A computation of the wave limit is obtained through the canonical model of the perturbation situation. A transformation results from the space $\mathscr{L}(\varphi)$ into the space $\mathscr{L}(\psi)$ when $\varphi(z)$ and $\psi(z)$ are reciprocals.

THEOREM 1. *If $\mathscr{L}(\varphi)$ and $\mathscr{L}(\psi)$ are spaces such that*

$$\varphi(z)\, \psi(z) = 1 = \psi(z)\, \varphi(z),$$

then the transformation which takes $f(z)$ into $\varphi(z) f(z)$ is an isometry of $\mathcal{L}(\psi)$ onto $\mathcal{L}(\varphi)$. A transformation $W(\psi, \varphi)$ of $\mathcal{L}(\varphi)$ into $\mathcal{L}(\psi)$ exists such that the identity

$$\langle f(z), \varphi(z) h(z)\rangle_{\mathcal{L}(\varphi)} - \langle g(z), h(z)\rangle_{\mathcal{L}(\psi)} = 2\langle f(z), h(z)\rangle$$

holds for every square summable element $h(3)$ of $\mathcal{L}(\psi)$ whenever $f(z)$ is a square summable element of $\mathcal{L}(\varphi)$ and $W(\psi, \varphi)$ takes $f(z)$ into $g(z)$. The transformation is bounded by one, it vanishes on the orthogonal complement in $\mathcal{L}(\varphi)$ of the square summable elements of $\mathcal{L}(\varphi)$, and its range is contained in the closure in $\mathcal{L}(\psi)$ of the square summable elements of $\mathcal{L}(\psi)$. The transformation takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

The transformation $W(\psi, \varphi)$ will be referred to as the wave limit of $\mathcal{L}(\varphi)$ into $\mathcal{L}(\psi)$. Its adjoint is the wave limit $W(\varphi, \psi)$ of $\mathcal{L}(\psi)$ into $\mathcal{L}(\varphi)$. The problem is to show that these transformations are nontrivial. When they are well-behaved, they are partial isometries with the range of $W(\psi, \varphi)$ equal to the closure in $\mathcal{L}(\psi)$ of the square summable elements of $\mathcal{L}(\psi)$ and the range of $W(\varphi, \psi)$ equal to tle closure in $\mathcal{L}(\varphi)$ of the square summable elements of $\mathcal{L}(\varphi)$.

The perturbation theory of unitary transformations is a limiting case of a perturbation theory of nonunitary transformations. A basic concept in the theory is the overlapping space of a space $\mathcal{H}(B)$.

A necessary and sufficient condition for a space $\mathcal{H}(B)$ to be contained isometrically in $\mathcal{C}(z)$ is that it contain no nonzero element of the form $B(z) f(z)$ with $f(z)$ in $\mathcal{C}(z)$. The overlapping space $\mathcal{L}$ of a space $\mathcal{H}(B)$ is the set of elements $f(z)$ of $\mathcal{C}(z)$ such that $B(z) f(z)$ belongs to $\mathcal{H}(B)$. The overlapping space is a Hilbert space in the norm

$$\| f(z)\|_{\mathcal{L}}^2 = \| f(z)\|^2 + \| B(z) f(z)\|_B^2 .$$

The series $[f(z) - f(0)]/z$ belongs to $\mathcal{L}$ whenever $f(z)$ belongs to $\mathcal{L}$, and its norm does not exceed the norm of $f(z)$. The transformation which takes $f(z)$ into $[f(z) - f(0)]/z$ has an isometric adjoint in $\mathcal{L}$. These conditions imply that the overlapping space $\mathcal{L}$ is isometrically equal to a space $\mathcal{L}(\theta)$. The space is contained in $\mathcal{C}(z)$ and the inclusion does not increase norms.

Another basic concept in the perturbation theory of nonunitary transformations is the extension space $\mathcal{D}(B)$ of a space $\mathcal{H}(B)$. This is a Hilbert space whose elements are pairs of power series with vector coefficients. A pair $(f(z), g(z))$ belongs to $\mathcal{D}(B)$ if $f(z)$ belongs to $\mathcal{H}(B)$ and if $g(z) = \sum a_n z^n$ where

$$z^{n+1} f(z) - B(z)(a_0 z^n + \cdots + a_n)$$

belongs to $\mathcal{H}(B)$ for every nonnegative integer n and if the sequence of numbers

$$\| z^{n+1} f(z) - B(z)(a_0 z^n + \cdots + a_n)\|_B^2 + | a_0 |^2 + \cdots + | a_n |^2$$

is bounded. The sequence is nondecreasing. Its limit is taken as the definition of $\|(f(z), g(z))\|^2_{\mathscr{D}(B)}$.

The space $\mathscr{D}(B)$ is a Hilbert space. The transformation which takes $(f(z), g(z))$ into $f(z)$ is a partial isometry of $\mathscr{D}(B)$ onto $\mathscr{H}(B)$. The identity

$$\|([f(z)-f(0)]/z, zg(z) - B^*(z)f(0))\|^2_{\mathscr{D}(B)} = \|(f(z), g(z))\|^2_{\mathscr{D}(B)} - |f(0)|^2$$

holds for every element $(f(z), g(z))$ of the space. The pair

$$([1 - B(z)\bar{B}(w)]\, c/(1 - z\bar{w}), [B^*(z) - \bar{B}(w)]\, c/(z - \bar{w}))$$

belongs to the space for every vector c when $|w| < 1$. The identity

$$\bar{c}f(w)$$
$$= \langle (f(z), g(z)), ([1 - B(z)\bar{B}(w)]\, c/(1 - z\bar{w}), [B^*(z) - \bar{B}(w)]\, c/(z - \bar{w}))\rangle_{\mathscr{D}(B)}$$

holds for every element $(f(z), g(z))$ of the space. The transformation which takes $(f(z), g(z))$ into $(g(z), f(z))$ is an isometry of $\mathscr{D}(B)$ onto $\mathscr{D}(B^*)$.

A relation holds between the extension space of a space $\mathscr{H}(B)$ and the extension space of its overlapping space. If $\mathscr{L}(\theta)$ is the overlapping space of $\mathscr{H}(B)$, then $\mathscr{E}(\theta)$ is the set of pairs $(f(z), g(z))$ of elements of $\mathscr{C}(z)$ such that $(B(z)f(z), -g(z))$ belongs to $\mathscr{D}(B)$. The identity

$$\|(f(z), g(z))\|^2_{\mathscr{E}(\theta)} = \|f(z)\|^2 + \|(B(z)f(z), -g(z))\|^2_{\mathscr{D}(B)}$$

holds for every element $(f(z), g(z))$ of $\mathscr{E}(\theta)$.

A perturbation situation is created when two spaces $\mathscr{H}(B)$ are suitably related. If $\mathscr{H}(A)$ and $\mathscr{H}(B)$ are spaces such that

$$B(z) = [\lambda - A(z)]/[1 - \bar{\lambda}A(z)]$$

for a number λ, $|\lambda| < 1$, then the transformation which takes $(f(z), g(z))$ into $([1 - \bar{\lambda}B(z)]f(z), -[1 - \lambda B^*(z)]\, g(z))$ takes $\mathscr{D}(A)$ onto $\mathscr{D}(B)$. The identity

$$(1 - \lambda\bar{\lambda})\|(f(z), g(z))\|^2_{\mathscr{D}(A)} = \|([1 - \bar{\lambda}B(z)]f(z), -[1 - \lambda B^*(z)]\, g(z))\|^2_{\mathscr{D}(B)}$$

holds for every element $(f(z), g(z))$ of $\mathscr{D}(A)$.

This result allows the transformation which takes $(f(z), g(z))$ into $([f(z) - f(0)]/z, zg(z) - B^*(z)f(0))$ in the space $\mathscr{D}(B)$ to be regarded as a perturbation of the transformation which takes $(f(z), g(z))$ into $([f(z) - f(0)]/z, zg(z) - A^*(z)f(0))$ in the space $\mathscr{D}(A)$. A fundamental identity in this perturbation situation states that

$$\begin{aligned}
&\bar{u}(0) f(0) \\
&\quad = \langle (f(z), g(z)), ([1 - \bar{\lambda} A(z)]\, u(z), -[1 - \lambda A^*(z)]\, v(z))\rangle_{\mathscr{D}(A)} \\
&\qquad - \langle ([f(z) - f(0)]/z, zg(z) - A^*(z) f(0)), \\
&\qquad \times ([1 - \bar{\lambda} A(z)]\, [u(z) - u(0)]/z, -[1 - \lambda A^*(z)]\, [zv(z) - B^*(z)\, u(0)])\rangle_{\mathscr{D}(A)}
\end{aligned}$$

for all elements $(f(z), g(z))$ of $\mathscr{D}(A)$ and $(u(z), v(z))$ of $\mathscr{D}(B)$.

The generalization of the wave limit in the perturbation theory of nonunitary transformations is a transformation of the overlapping space of $\mathscr{H}(A)$ onto the overlapping space of $\mathscr{H}(B)$.

THEOREM 2. *If $\mathscr{H}(A)$ and $\mathscr{H}(B)$ are spaces such that*

$$B(z) = [\lambda - A(z)]/[1 - \bar{\lambda} A(z)]$$

for a number λ, $|\lambda| < 1$, then the transformation which takes $f(z)$ into $[1 - \bar{\lambda} B(z)]\, f(z)$ takes $\mathscr{H}(A)$ onto $\mathscr{H}(B)$ and the identity

$$(1 - \lambda\bar{\lambda})\, \| f(z)\|_A^2 = \|[1 - \bar{\lambda} B(z)]\, f(z)\|_B^2$$

holds for every element $f(z)$ of $\mathscr{H}(A)$. If $\mathscr{L}(\varphi)$ is the overlapping space of $\mathscr{H}(A)$ and if $\mathscr{L}(\psi)$ is the overlapping space of $\mathscr{H}(B)$, then a transformation $W(\psi, \varphi)$ of $\mathscr{L}(\varphi)$ onto $\mathscr{L}(\psi)$ exists such that the identity

$$\langle g(z), h(z)\rangle + \langle f(z), [1 - \bar{\lambda} B(z)]\, h(z)\rangle = 0$$

holds for every element $h(z)$ of $\mathscr{C}(z)$ whenever $W(\psi, \varphi)$ takes $f(z)$ into $g(z)$. If $W(\psi, \varphi)$ takes $f(z)$ into $g(z)$, then

$$(1 - \lambda\bar{\lambda})\, \| f(z)\|_{\mathscr{L}(\varphi)}^2 = \| g(z)\|_{\mathscr{L}(\psi)}^2$$

and the identity

$$\langle [1 - \bar{\lambda} B(z)]\, A(z) f(z), h(z)\rangle_B - \langle B(z)\, g(z), h(z)\rangle_B = \lambda \langle f(z), h(z)\rangle$$

holds for every element $h(z)$ of $\mathscr{H}(B)$. The transformation takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

The transformation $W(\psi, \varphi)$ will be called the wave limit of $\mathscr{L}(\varphi)$ onto $\mathscr{L}(\psi)$.

A mixed perturbation situation is created in the context used to construct spaces $\mathscr{L}(\varphi)$ from spaces $\mathscr{H}(B)$. If $\mathscr{L}(\varphi)$ and $\mathscr{H}(B)$ are spaces such that

$$B(z) = [1 - \varphi(z)]/[1 + \varphi(z)],$$

then the transformation which takes $(f(z), g(z))$ into $([f(z) - f(0)]/z, zg(z) + f(0))$ in $\mathscr{E}(\varphi)$ and the transformation which takes $(f(z), g(z))$ into

$$([f(z) - f(0)]/z, zg(z) - B^*(z) f(0))$$

in $\mathscr{D}(B)$ can be regarded as perturbations of each other. A fundamental identity of this perturbation theory states that

$$\begin{aligned}\bar{u}(0)f(0) = &\langle([1+B(z)]f(z), -[1+B^*(z)]g(z)), (u(z), v(z))\rangle_{\mathscr{D}(B)} \\ &- \langle([1+B(z)][f(z)-f(0)]/z, -[1+B^*(z)][zg(z)+f(0)]), \\ &\times ([u(z)-u(0)]/z, zv(z)-B^*(z)u(0))\rangle_{\mathscr{D}(B)}\end{aligned}$$

for all elements $(f(z), g(z))$ of $\mathscr{E}(\varphi)$ and $(u(z), v(z))$ of $\mathscr{D}(B)$.

The analog of the wave limit is a partial isometry of $\mathscr{L}(\varphi)$ into the overlapping space of $\mathscr{H}(B)$.

THEOREM 3. *If $\mathscr{L}(\varphi)$ and $\mathscr{H}(B)$ are spaces such that*

$$B(z) = [1-\varphi(z)]/[1+\varphi(z)],$$

then the transformation which takes $f(z)$ into $[1+B(z)]f(z)$ is an isometry of $\mathscr{L}(\varphi)$ onto $\mathscr{H}(B)$. If $\mathscr{L}(\theta)$ is the overlapping space of $\mathscr{H}(B)$, a partially isometric transformation $W(\theta, \varphi)$ of $\mathscr{L}(\varphi)$ into $\mathscr{L}(\theta)$ exists with these properties: The transformation takes $f(z)$ into $g(z)$ whenever $f(z)$ is a square summable element of $\mathscr{L}(\varphi)$ and $g(z)$ is the element of $\mathscr{C}(z)$ such that

$$\langle g(z), h(z)\rangle = \langle f(z), [1+B(z)]h(z)\rangle$$

for every element $h(z)$ of $\mathscr{C}(z)$. The norm of $g(z)$ in $\mathscr{L}(\theta)$ is equal to the norm of $f(z)$ in $\mathscr{L}(\varphi)$ whenever $f(z)$ and $g(z)$ are so related and the identity

$$\langle B(z)g(z), h(z)\rangle_B - \langle[1+B(z)]f(z), h(z)\rangle_B = -\langle f(z), h(z)\rangle,$$

holds for every element $h(z)$ of $\mathscr{H}(B)$. The kernel of the transformation is the orthogonal complement in $\mathscr{L}(\varphi)$ of the square summable elements of the space. The transformation takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$. The adjoint $W(\varphi, \theta)$ of $W(\theta, \varphi)$ is the transformation which takes $f(z)$ into $g(z)$ whenever $g(z)$ belongs to the closure in $\mathscr{L}(\varphi)$ of the square summable elements of the space and the identity

$$\langle g(z), h(z)\rangle_{\mathscr{L}(\varphi)} - \langle B(z)f(z), [1+B(z)]h(z)\rangle_B = \langle f(z), h(z)\rangle$$

holds for every square summable element $h(z)$ of $\mathscr{L}(\varphi)$. The transformation is a partial isometry which takes $[f(z)-f(0)]/z$ into $[g(z)-g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

The transformation $W(\theta, \varphi)$ will be called the wave limit of $\mathscr{L}(\varphi)$ into $\mathscr{L}(\theta)$ and its adjoint $W(\varphi, \theta)$ will be called the wave limit of $\mathscr{L}(\theta)$ into $\mathscr{L}(\varphi)$. The mixed wave limits are related to the wave limits in the perturbation theory of unitary transformations.

THEOREM 4. *Let* $\mathcal{L}(\varphi)$, $\mathcal{L}(\psi)$, *and* $\mathcal{H}(B)$ *be spaces such that*

$$\varphi(z) = [1 - B(z)]/[1 + B(z)] \qquad \textit{and} \qquad \psi(z) = [1 + B(z)]/[1 - B(z)],$$

and let $\mathcal{L}(\theta)$ *be the overlapping space of* $\mathcal{H}(B)$. *Then the identity*

$$-W(\psi, \varphi) = W(\psi, \theta)\, W(\theta, \varphi)$$

holds for the wave limits $W(\psi, \varphi)$ *of* $\mathcal{L}(\varphi)$ *into* $\mathcal{L}(\psi)$, $W(\theta, \varphi)$ *of* $\mathcal{L}(\varphi)$ *into* $\mathcal{L}(\theta)$, *and* $W(\psi, \theta)$ *of* $\mathcal{L}(\theta)$ *into* $\mathcal{L}(\psi)$.

The study of the wave limit $W(\psi, \varphi)$ is thereby reduced to the determination of the ranges of the partially isometric wave limits $W(\theta, \varphi)$ and $W(\theta, \psi)$, which have an elementary characterization.

THEOREM 5. *Let* $\mathcal{L}(\varphi)$ *and* $\mathcal{H}(B)$ *be spaces related as in Theorem* 3, *and let* $W(\theta, \varphi)$ *be the wave limit of* $\mathcal{L}(\varphi)$ *into the overlapping space* $\mathcal{L}(\theta)$ *of* $\mathcal{H}(B)$. *Then* $\mathcal{L}(\theta)$ *is contained in* $\mathcal{L}(2 + 2B)$ *and the inclusion does not increase norms. All polynomials with vector coefficients belong to* $\mathcal{L}(2 + 2B)$. *The orthogonal complement in* $\mathcal{L}(2 + 2B)$ *of these polynomials coincides with the orthogonal complement in* $\mathcal{L}(\theta)$ *of the range of* $W(\theta, \varphi)$ *and is the set of elements of* $\mathcal{L}(\theta)$ *which have the same norm in* $\mathcal{L}(\theta)$ *as in* $\mathcal{L}(2 + 2B)$. *The extension space* $\mathcal{E}(\theta)$ *of* $\mathcal{L}(\theta)$ *is contained in the extension space* $\mathcal{E}(2 + 2B)$ *of* $\mathcal{L}(2 + 2B)$ *and the inclusion does not increase norms. A necessary and sufficient condition that an element* $f(z)$ *of* $\mathcal{L}(\theta)$ *have the same norm in* $\mathcal{L}(\theta)$ *as in* $\mathcal{L}(2 + 2B)$ *is that* $(f(z), g(z))$ *has the same norm in* $\mathcal{E}(\theta)$ *as in* $\mathcal{E}(2 + 2B)$ *for some element* $g(z)$ *of* $\mathcal{L}(\theta^*)$.

Another characterization of the range is obtained using the boundary value function theory for analytic functions which are represented by square summable power series. If $f(z)$ belongs to $\mathcal{C}(z)$, an essentially unique measurable vector valued function $f(e^{it})$ of t in the reals modulo 2π exists such that

$$2\pi \|f(z)\|^2 = \int_0^{2\pi} |f(e^{it})|^2 \, dt,$$

$$2\pi f(w) = \int_0^{2\pi} f(e^{it})\, (1 - e^{-it} w)^{-1} \, dt,$$

and

$$0 = \int_0^{2\pi} f(e^{it})\, (e^{-it} - w)^{-1} \, dt,$$

when $|w| < 1$. A boundary value function $f(e^{it})$ will be defined more generally for an element $(f(z), g(z))$ of $\mathcal{E}(1)$ so that

$$2\pi \|f(z)\|^2 + 2\pi \|g(z)\|^2 = \int^{2\pi} |f(e^{it})|^2 \, dt$$

and

$$2\pi f(w) = \int_0^{2\pi} f(e^{it})\,(1 - e^{-it}w)^{-1}\,dt$$

and

$$2\pi g(w) = \int_0^{2\pi} f(e^{it})\,(e^{-it} - w)^{-1}\,dt,$$

when $|w| < 1$.

Hilbert spaces of vectors are used in the study of boundary value functions. If P is a nonnegative operator, let $\mathscr{C}(P)$ be the unique Hilbert space of vectors such that Pa belongs to the space for every vector a and such that the identity

$$\bar{a}b = \langle b, Pa\rangle_P$$

holds for every element b of the space.

If $B(z)$ is a power series with operator coefficients such that a space $\mathscr{H}(B)$ exists, a measurable operator valued function $B(e^{it})$ of t in the reals modulo 2π exists such that $B(e^{it})f(e^{it})$ is the boundary value function of $B(z)f(z)$ whenever $f(z)$ is an element of $\mathscr{C}(z)$ with boundary value function $f(e^{it})$. The boundary value function $B(e^{it})$ is essentially unique and its values can be chosen bounded by one. The boundary value function is used to construct the overlapping space of $\mathscr{H}(B)$.

THEOREM 6. *If $\mathscr{H}(B)$ is a given space with overlapping space $\mathscr{L}(\theta)$, a necessary and sufficient condition that an element $(f(z), g(z))$ of $\mathscr{E}(1)$ belong to $\mathscr{E}(\theta)$ is that its boundary value function $f(e^{it})$ satisfy the condition*

$$\int_0^{2\pi} \|f(e^{it})\|^2_{\mathscr{C}(1-\bar{B}(e^{it})B(e^{it}))}\,dt < \infty.$$

In this case the integral is equal to

$$2\pi\,\|(f(z), g(z))\|^2_{\mathscr{E}(\theta)}\,.$$

A necessary and sufficient condition that $(f(z), g(z))$ belong to $\mathscr{E}(2 + 2B)$ is that

$$\int_0^{2\pi} \|f(e^{it})\|^2_{\mathscr{C}(2+B(e^{it})+\bar{B}(e^{it}))}\,dt < \infty.$$

In this case the integral is equal to

$$2\pi\,\|(f(z), g(z))\|^2_{\mathscr{E}(2+2B)}\,.$$

A decomposition is used to compute the range of the wave limit from a knowledge of boundary value functions.

THEOREM 7. *If B is an operator which is bounded by one, the space $\mathscr{C}(1 - \bar{B}B)$ is contained in the space $\mathscr{C}(2 + B + \bar{B})$ and the inclusion does not increase norms. The operator $1 + \bar{B}$ is a transformation of $\mathscr{C}$ into $\mathscr{C}(2 + B + \bar{B})$ which does not increase norms. The orthogonal complement of the range of the transformation in $\mathscr{C}(2 + B + \bar{B})$ is contained in $\mathscr{C}(1 - \bar{B}B)$ and is the set of elements of $\mathscr{C}(1 - \bar{B}B)$ which have the same norm in $\mathscr{C}(1 - \bar{B}B)$ as in $\mathscr{C}(2 + B + \bar{B})$.*

Nonzero elements of $\mathscr{C}(1 - \bar{B}B)$ may exist which have the same norm in $\mathscr{C}(1 - \bar{B}B)$ as in $\mathscr{C}(2 + B + \bar{B})$. Examples are found when $\mathscr{C}$ is the Hilbert space of square summable power series with complex coefficients. Let $\varphi(z)$ be a power series with complex coefficients which represents a function with nonnegative real part in the unit disk. Then multiplication by $A(z) = [1 - \varphi(z)]/[1 + \varphi(z)]$ is a transformation in the space of square summable power series with complex coefficients which is bounded by one. Let B be the adjoint of multiplication by $A(z)$ as a transformation in this space. Then $\mathscr{C}(1 - \bar{B}B)$ is isometrically equal to the space which is called $\mathscr{H}(A)$ in the notation for square summable power series with complex coefficients. The elements of $\mathscr{C}(1 - \bar{B}B)$ which have the same norm in $\mathscr{C}(1 - \bar{B}B)$ as in $\mathscr{C}(2 + B + \bar{B})$ are the orthogonal complement in $\mathscr{C}(1 - \bar{B}B)$ of the elements of the space of the form $A(z) f(z)$ for a square summable power series $f(z)$ with complex coefficients.

A computation of this space is made using the Poisson representation of functions which are analytic and have nonnegative real part in the unit disk. Let $\mu(t)$ be a nondecreasing function of t, $0 \leqslant t \leqslant 2\pi$, such that

$$\operatorname{Re} \varphi(w) = (1 - |w|^2)/2\pi \int_0^{2\pi} d\mu(t)/|1 - e^{-it}w|^2$$

when $|w| < 1$. Let $\mu(t) = \mu_0(t) + \mu_1(t)$, where $\mu_0(t)$ and $\mu_1(t)$ are nondecreasing functions of t, $0 \leqslant t \leqslant 2\pi$, such that $d\mu_0(t)$ is singular with respect to Lebesgue measure and $d\mu_1(t)$ is absolutely continuous with respect to Lebesgue measure. Let $\varphi(z) = \varphi_0(z) + \varphi_1(z)$ for power series with complex coefficients representing functions with nonnegative real parts in the unit disk given by the last formula with subscript k on $\varphi(z)$ and $\mu(t)$ for k equal to zero and one. Let $\mathscr{H}$, $\mathscr{H}_0$, and $\mathscr{H}_1$ be the Hilbert spaces of constants which are denoted $\mathscr{L}(\varphi)$, $\mathscr{L}(\varphi_0)$, and $\mathscr{L}(\varphi_1)$ in the notation for square summable power series with complex coefficients. Then $\mathscr{H}_0$ and $\mathscr{H}_1$ are contained isometrically in $\mathscr{H}$ and form an orthogonal decomposition of the space. Multiplication by $1 + A(z)$ is an isometric transformation of $\mathscr{H}$ onto $\mathscr{C}(1 - \bar{B}B)$. It takes $\mathscr{H}_0$ isometrically onto the set of elements of $\mathscr{C}(1 - \bar{B}B)$ which have the same norm in $\mathscr{C}(1 - \bar{B}B)$ as in $\mathscr{C}(2 + B + \bar{B})$.

Despite this example it is a common occurrence that no nonzero element of $\mathscr{C}(1 - \bar{B}B)$ has the same norm in $\mathscr{C}(1 - \bar{B}B)$ as in $\mathscr{C}(2 + B + \bar{B})$. In applications to perturbation theory, this means that the mixed wave limit of Theorem 3 frequently has the full overlapping space $\mathscr{L}(\theta)$ as its range. Such results for mixed wave limits are obtained from their relation to the wave limits for nonunitary operators. Some form of dominated convergence is needed for the limiting procedure involved to be well-behaved. A hypothesis is offered which produces the desired effect.

THEOREM 8. *Let $\mathscr{L}(\varphi)$ and $\mathscr{H}(B)$ be spaces related as in Theorem 3, let $\mathscr{L}(\theta)$ be the overlapping space of $\mathscr{H}(B)$, and let λ be a given number, $0 < \lambda < 1$. Let $\mathscr{L}(\theta_\lambda)$ be the overlapping space of $\mathscr{H}(B_\lambda)$,*

$$B_\lambda(z) = [B(z) + \lambda]/[1 + \lambda B(z)].$$

Assume that the inequality

$$\|f(z)\|^2 - 2\,\|B(z)f(z)\|^2 + \|B(z)^2 f(z)\|^2 \geqslant -(1-\lambda)^{-1}\,\|[1 + B(z)]^2 f(z)\|^2$$

holds for every element $f(z)$ of $\mathscr{C}(z)$. Then the inequality

$$\|f(z)\|^2_{\mathscr{L}(1+\varphi)} \leqslant (1-\lambda^2)\,\|f(z)\|^2_{\mathscr{L}(\theta_\lambda)}$$

holds for every element $f(z)$ of $\mathscr{L}(\theta_\lambda)$.

The hypothesis is satisfied, for example, if the inequality $\|B(z)f(z)\| \geqslant \|B^*(z)\,f(z)\|$ holds for every element $f(z)$ of $\mathscr{C}(z)$. When the hypothesis is satisfied, the norm of $\mathscr{L}(1+\varphi)$ becomes an effective majorant for the limiting behavior of nonunitary wave limits.

THEOREM 9. *Let $\mathscr{L}(\varphi)$ and $\mathscr{H}(B)$ be spaces related as in Theorem 3, and let $\mathscr{L}(\theta)$ be the overlapping space of $\mathscr{H}(B)$. Assume that a number λ, $0 < \lambda < 1$, exists such that the inequality*

$$\|f(z)\|^2 - 2\,\|B(z)f(z)\|^2 + \|B(z)^2 f(z)\|^2 \geqslant -(1-\lambda)^{-1}\,\|[1 + B(z)]^2 f(z)\|^2$$

holds for every element $f(z)$ of $\mathscr{C}(z)$. Then the wave limit $W(\theta, \varphi)$ of $\mathscr{L}(\varphi)$ into $\mathscr{L}(\theta)$ takes $\mathscr{L}(\varphi)$ onto $\mathscr{L}(\theta)$.

Computable examples illustrating the theory are obtained when $\varphi(z)$ is a rational function.

THEOREM 10. *If $\varphi(0)$ and U are operators, $\varphi(0)$ with nonnegative real part, such that*

$$\varphi(0) + \bar{\varphi}(0) = U[\varphi(0) + \bar{\varphi}(0)]\,\bar{U},$$

then a space $\mathscr{L}(\varphi)$ exists,

$$\varphi(z) = (1 - zU)^{-1}\,[\varphi(0) + zU\bar{\varphi}(0)],$$

and the identity $[f(z) - f(0)]/z = Uf(z)$ *holds for every element* $f(z)$ *of the space.*

Corresponding examples are obtained when $B(z)$ is a rational function.

THEOREM 11. *If* $B(0)$ *and* S *are operators,* $B(0)$ *bounded by one, such that*

$$1 - B(0)\bar{B}(0) = S[1 - \bar{B}(0)B(0)]\bar{S},$$

then a space $\mathscr{H}(B)$ *exists,*

$$B(z) = [1 - zS\bar{B}(0)]^{-1}[B(0) - zS],$$

and the identity $[f(z) - f(0)]/z = S\bar{B}(0)f(0)$ *holds for every element* $f(z)$ *of the space. The identity* $[S - zB(0)]f(z) = Sf(0)$ *holds for every element* $f(z)$ *of* $\mathscr{H}(B^*)$. *The identity* $[B(0) - Sz]f(z) = B(0)f(0)$ *holds for every element* $f(z)$ *of the overlapping space* $\mathscr{L}(\theta)$ *of* $\mathscr{H}(B)$.

The spaces $\mathscr{L}(\varphi)$ and $\mathscr{H}(B)$ so obtained are properly related to each other.

THEOREM 12. *If* $\varphi(0)$ *and* U *are operators which satisfy the hypotheses of Theorem* 10, *then*

$$B(0) = [1 - \varphi(0)]/[1 + \varphi(0)]$$

and

$$S = [1 - \varphi(0)]^{-1}U[1 + \bar{\varphi}(0)]$$

are operators which satisfy the hypotheses of Theorem 11, *and*

$$B(z) = [1 - \varphi(z)]/[1 + \varphi(z)].$$

In such examples multiplication by $B^*(z)$ is isometric in $\mathscr{C}(z)$ under a hypothesis natural to the situation [4]. It is that the equation $[S\bar{B}(0) - z]f(z) = S\bar{B}(0)f(0)$ has no nonzero solution $f(z)$ in $\mathscr{C}(z)$. The condition is satisfied for example if $S\bar{B}(0)$ is bounded by one.

Proof of Theorem 1. If $f(z) = a_0 + a_1z + a_2z^2 + \cdots$ and $g(z) = b_0 + b_1z + b_2z^2 + \cdots$ are power series with vector coefficients, define

$$\langle f(z), g(z)\rangle_n = \bar{b}_0a_0 + \cdots + \bar{b}_na_n.$$

If $(f(z), g(z))$ is an element of $\mathscr{E}(\varphi)$, define elements $(f_n(z), g_n(z))$ of $\mathscr{E}(\varphi)$ inductively for every nonnegative integer n so that $f_0(z) = f(z)$ and $g_0(z) = g(z)$ and so that $f_{n+1}(z) = [f_n(z) - f_n(0)]/z$ and $g_{n+1}(z) = zg_n(z) + f_n(0)$, for every n. If $(u(z), v(z))$ is an element of $\mathscr{E}(\psi)$, define elements $(u_n(z), v_n(z))$ of $\mathscr{E}(\psi)$ inductively by the same method. Since

$$\begin{aligned}2\bar{u}_n(0)f_n(0) &= \langle(f_n(z), g_n(z)), (\varphi(z)u_n(z), -\varphi^*(z)v_n(z))\rangle_{\mathscr{E}(\varphi)}\\ &\quad - \langle(f_{n+1}(z), g_{n+1}(z)), (\varphi(z)u_{n+1}(z), -\varphi^*(z)v_{n+1}(z))\rangle_{\mathscr{E}(\varphi)}\end{aligned}$$

for every index n,

$$2\langle f(z), u(z)\rangle_n = \langle (f(z), g(z)), (\varphi(z)\, u(z), -\varphi^*(z)\, v(z))\rangle_{\mathscr{E}(\varphi)} - \langle (\psi(z) f_{n+1}(z), -\psi^*(z)\, g_{n+1}(z)), (u_{n+1}(z), v_{n+1}(z))\rangle_{\mathscr{E}(\varphi)} .$$

It follows that an isometric transformation S_n of $\mathscr{E}(\varphi)$ into $\mathscr{E}(\psi)$ exists such that the identity

$$2\langle f(z), u(z)\rangle_n = \langle (f(z), g(z)), (\varphi(z)\, u(z), -\varphi^*(z)\, v(z))\rangle_{\mathscr{E}(\varphi)} - \langle (h(z), k(z)), (u(z), v(z))\rangle_{\mathscr{E}(\psi)}$$

holds for every element $(u(z), v(z))$ of $\mathscr{E}(\psi)$ whenever S_n takes $(f(z), g(z))$ into $(h(z), k(z))$. In the case that $f(z) = 0$, then $u(z) = 0$ and $k(z) = -\psi^*(z)\, g(z)$. The stated results are derived in an obvious way in the limit of large n.

Proof of Theorem 2. Since

$$[1 - \bar{\lambda} A(z)]\,[1 - \bar{\lambda} B(z)] = 1 - \lambda\bar{\lambda},$$

where $1 - \lambda\bar{\lambda}$ is positive, a unique Hilbert space $\mathscr{H}$ of square summable power series exists such that the transformation which takes $f(z)$ into $[1 - \bar{\lambda} B(z)]\, f(z)$ takes $\mathscr{H}(A)$ onto $\mathscr{H}$ and such that the identity

$$(1 - \lambda\bar{\lambda})\, \| f(z)\|_A^2 = \|[1 - \bar{\lambda} B(z)]\, f(z)\|_{\mathscr{H}}^2$$

holds for every element $f(z)$ of $\mathscr{H}(A)$. It follows that the identity

$$(1 - \lambda\bar{\lambda})\, \langle f(z), g(z)\rangle_A = \langle [1 - \bar{\lambda} B(z)]\, f(z), [1 - \bar{\lambda} B(z)]\, g(z)\rangle_{\mathscr{H}}$$

holds for all elements $f(z)$ and $g(z)$ of $\mathscr{H}(A)$. Since

$$[1 - \bar{\lambda} B(z)]\,[1 - A(z)\, \bar{A}(w)]\,[1 - \lambda \bar{B}(w)] = (1 - \lambda\bar{\lambda})\,[1 - B(z)\, \bar{B}(w)],$$

the series $[1 - B(z)\, \bar{B}(w)]\, c/(1 - z\bar{w})$ belongs to $\mathscr{H}$ for every vector c when $|w| < 1$ and the identity

$$\bar{c} f(w) = \langle f(z), [1 - B(z)\, \bar{B}(w)]\, c/(1 - z\bar{w})\rangle_{\mathscr{H}}$$

holds for every element $f(z)$ of $\mathscr{H}$. It follows that $\mathscr{H}$ is isometrically equal to $\mathscr{H}(B)$.

If $f(z)$ belongs to the overlapping space $\mathscr{L}(\varphi)$ of $\mathscr{H}(A)$, consider the unique element $g(z)$ of $\mathscr{C}(z)$ such that the identity

$$\langle g(z), h(z)\rangle + \langle f(z), [1 - \bar{\lambda} B(z)]\, h(z)\rangle = 0$$

holds for every element $h(z)$ of $\mathscr{C}(z)$. By the theory of minimal decompositions of an element of $\mathscr{C}(z)$ with respect to $\mathscr{H}(B)$, $\lambda f(z) - B(z)f(z) - B(z)g(z)$ belongs to $\mathscr{H}(B)$ and the identity

$$\langle \lambda f(z) - B(z)f(z) - B(z)g(z), h(z)\rangle_B = \lambda\langle f(z), h(z)\rangle$$

holds for every element $h(z)$ of $\mathscr{H}(B)$. Since $A(z)f(z)$ belongs to $\mathscr{H}(A)$,

$$[\lambda - B(z)]f(z) = [1 - \bar{\lambda}B(z)]A(z)f(z)$$

belongs to $\mathscr{H}(B)$. So $B(z)g(z)$ belongs to $\mathscr{H}(B)$ and $g(z)$ belongs to the overlapping space $\mathscr{L}(\psi)$ of $\mathscr{H}(B)$. Also

$$\begin{aligned}
&\|g(z)\|^2_{\mathscr{L}(\psi)} \\
&= \|g(z)\|^2 + \|B(z)g(z)\|^2_B \\
&= \|g(z)\|^2 + \|[\lambda f(z) - B(z)f(z) - B(z)g(z)] - [1 - \bar{\lambda}B(z)]A(z)f(z)\|^2_B \\
&= \|g(z)\|^2 + \lambda\langle f(z), \lambda f(z) - B(z)f(z) - B(z)g(z)\rangle \\
&\quad - \lambda\langle f(z), [\lambda - B(z)]f(z)\rangle - \bar{\lambda}\langle[\lambda - B(z)]f(z), f(z)\rangle \\
&\quad + (1 - \lambda\bar{\lambda})\|A(z)f(z)\|^2_A \\
&= \|g(z)\|^2 - \lambda\langle f(z), B(z)g(z)\rangle + \bar{\lambda}\langle B(z)f(z), g(z)\rangle \\
&\quad - \lambda\bar{\lambda}\|f(z)\|^2 + (1 - \lambda\bar{\lambda})\|A(z)f(z)\|^2_A \\
&= (1 - \lambda\bar{\lambda})\|f(z)\|^2 + (1 - \lambda\bar{\lambda})\|A(z)f(z)\|^2_A \\
&= (1 - \lambda\bar{\lambda})\|f(z)\|^2_{\mathscr{L}(\psi)}.
\end{aligned}$$

A similar argument will show that if $g(z)$ belongs to $\mathscr{L}(\psi)$, then the unique element $f(z)$ of $\mathscr{C}(z)$ such that the identity

$$\langle g(z), h(z)\rangle + \langle f(z), [1 - \bar{\lambda}B(z)]h(z)\rangle = 0$$

holds for every element $h(z)$ of $\mathscr{C}(z)$ belongs to $\mathscr{L}(\varphi)$. So a unique transformation $W(\psi, \varphi)$ of $\mathscr{L}(\varphi)$ onto $\mathscr{L}(\psi)$ exists which takes $f(z)$ into $g(z)$ whenever the identity holds for every element $h(z)$ of $\mathscr{C}(z)$, and in this case

$$\|g(z)\|^2_{\mathscr{L}(\varphi)} = (1 - \lambda\bar{\lambda})\|f(z)\|^2_{\mathscr{L}(\psi)}.$$

The identity

$$\langle[1 - \bar{\lambda}B(z)]A(z)f(z), h(z)\rangle_B - \langle B(z)g(z), h(z)\rangle_B = \lambda\langle f(z), h(z)\rangle$$

holds when $h(z) = [1 - B(z)\bar{B}(w)]c/(1 - z\bar{w})$ for a vector c and a number w, $|w| < 1$, because it can be written

$$\langle g(z), \bar{B}(w)c/(1 - z\bar{w})\rangle + \langle f(z), [1 - \bar{\lambda}B(z)]\bar{B}(w)c/(1 - z\bar{w})\rangle = 0.$$

The identity follows by linearity and continuity for every element $h(z)$ of $\mathscr{H}(B)$ because $\mathscr{H}(B)$ is the closed span of such elements $[1 - B(z)\bar{B}(w)]\, c/(1 - z\bar{w})$.

Note that if $f(z)$ and $g(z)$ are elements of $\mathscr{C}(z)$ such that the identity

$$\langle g(z), h(z)\rangle + \langle f(z), [1 - \bar{\lambda} B(z)]\, h(z)\rangle = 0$$

holds for every element $h(z)$ of $\mathscr{C}(z)$, then the same identity holds with $h(z)$ replaced by $zh(z)$. An equivalent identity is then

$$\langle [g(z) - g(0)]/z, h(z)\rangle + \langle [f(z) - f(0)]/z, [1 - \bar{\lambda} B(z)]\, h(z)\rangle = 0.$$

Proof of Theorem 3. The argument is analogous to the proof of Theorem 2 except that now the isometry which takes $f(z)$ into $[1 + B(z)]\, f(z)$ of $\mathscr{L}(\varphi)$ onto $\mathscr{H}(B)$ exists by the construction of the space $\mathscr{L}(\varphi)$.

If $f(z)$ is a square summable element of $\mathscr{L}(\varphi)$, consider the unique element $g(z)$ of $\mathscr{C}(z)$ such that the identity

$$\langle g(z), h(z)\rangle = \langle f(z), [1 + B(z)]\, h(z)\rangle$$

holds for every element $h(z)$ of $\mathscr{C}(z)$. By the theory of minimal decompositions of an element of $\mathscr{C}(z)$ with respect to $\mathscr{H}(B)$, $f(z) + B(z) f(z) - B(z)\, g(z)$ belongs to $\mathscr{H}(B)$ and the identity

$$\langle f(z) + B(z) f(z) - B(z)\, g(z), h(z)\rangle_B = \langle f(z), h(z)\rangle$$

holds for every element $h(z)$ of $\mathscr{H}(B)$. Since $[1 + B(z)]\, f(z)$ belongs to $\mathscr{H}(B)$, $B(z)\, g(z)$ belongs to $\mathscr{H}(B)$ and $g(z)$ belongs to the overlapping space $\mathscr{L}(\theta)$ of $\mathscr{H}(B)$. Also

$$\begin{aligned}
&\| g(z)\|^2_{\mathscr{L}(\theta)} \\
&\quad = \| g(z)\|^2 + \| B(z)\, g(z)\|^2_B \\
&\quad = \| g(z)\|^2 + \|[f(z) + B(z) f(z) - B(z)\, g(z)] - [1 + B(z)]\, f(z)\|^2_B \\
&\quad = \| g(z)\|^2 + \langle f(z), f(z) + B(z) f(z) - B(z)\, g(z)\rangle \\
&\qquad\quad - \langle f(z), [1 + B(z)]\, f(z)\rangle - \langle [1 + B(z)]\, f(z), f(z)\rangle + \| f(z)\|^2_{\mathscr{L}(\varphi)} \\
&\quad = \| g(z)\|^2 - \langle f(z), B(z)\, g(z)\rangle - \langle B(z) f(z), f(z)\rangle - \| f(z)\|^2 + \| f(z)\|^2_{\mathscr{L}(\varphi)} \\
&\quad = \| f(z)\|^2_{\mathscr{L}(\varphi)} .
\end{aligned}$$

The identity

$$\langle B(z)\, g(z), h(z)\rangle_B - \langle [1 + B(z)]\, f(z), h(z)\rangle_B = -\langle f(z), h(z)\rangle$$

holds when $h(z) = [1 - B(z)\bar{B}(w)]c/(1 - z\bar{w})$ for a vector c and a number w, $|w| < 1$, because it can be written

$$\langle g(z), \bar{B}(w)c/(1 - z\bar{w})\rangle = \langle f(z), [1 + B(z)]\bar{B}(w)c/(1 - z\bar{w})\rangle.$$

The identity follows by linearity and continuity for all elements $h(z)$ of $\mathscr{H}(B)$ because $\mathscr{H}(B)$ is the closed span of such elements $[1 - B(z)\bar{B}(w)]c/(1 - z\bar{w})$.

If $f(z)$ and $g(z)$ are elements of $\mathscr{C}(z)$ such that

$$\langle g(z), h(z)\rangle = \langle f(z), [1 + B(z)]h(z)\rangle$$

for every element $h(z)$ of $\mathscr{C}(z)$, then

$$\langle [g(z) - g(0)]/z, h(z)\rangle = \langle [f(z) - f(0)]/z, [1 + B(z)]h(z)\rangle$$

for every element $h(z)$ of $\mathscr{C}(z)$.

The adjoint $W(\varphi, \theta)$ of $W(\theta, \varphi)$ is the partially isometric transformation whose kernel is the range of $W(\theta, \varphi)$ and which takes $f(z)$ into $g(z)$ whenever $g(z)$ is a square summable element of $\mathscr{L}(\varphi)$ and $W(\theta, \varphi)$ takes $g(z)$ into $f(z)$. Since the identity

$$\langle [1 + B(z)]g(z), h(z)\rangle_B - \langle B(z)f(z), h(z)\rangle_B = \langle g(z), f(z)\rangle$$

holds for every element $h(z)$ of $\mathscr{H}(B)$ and since the identity

$$\langle f(z), h(z)\rangle = \langle g(z), [1 + B(z)]h(z)\rangle$$

holds for every element $h(z)$ of $\mathscr{C}(z)$, the identity

$$\langle g(z), h(z)\rangle_{\mathscr{L}(\varphi)} - \langle B(z)f(z), [1 + B(z)]h(z)\rangle_B = \langle f(z), h(z)\rangle$$

holds for every square summable element $h(z)$ of $\mathscr{L}(\varphi)$.

The same identity holds with $g(z) = 0$ when $f(z)$ is orthogonal in $\mathscr{L}(\theta)$ to the range of $W(\theta, \varphi)$. For if $W(\theta, \varphi)$ takes $h(z)$ into $k(z)$, then

$$\langle f(z), k(z)\rangle_{\mathscr{L}(\theta)} = \langle f(z), k(z)\rangle + \langle B(z)f(z), B(z)k(z)\rangle_B = 0,$$

where

$$\langle f(z), k(z)\rangle = \langle [1 - B(z)]f(z), h(z)\rangle$$

and

$$\langle B(z)f(z), h(z) + B(z)h(z) - B(z)k(z)\rangle_B = \langle B(z)f(z), h(z)\rangle$$

by the theory of minimal decompositions of an element of $\mathscr{C}(z)$ with respect to $\mathscr{H}(B)$.

A straightforward calculation using the identity will show that $W(\varphi, \theta)$ takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$.

Proof of Theorem 4. Let $f(z)$ be a square summable element of $\mathscr{L}(\varphi)$, let $g(z)$ be the element of $\mathscr{L}(\theta)$ obtained from $f(z)$ under the action of $W(\theta, \varphi)$, and let $k(z)$ be the element of $\mathscr{L}(\psi)$ obtained from $f(z)$ under the action of $W(\psi, \varphi)$. By Theorem 1, the identity

$$\langle f(z), \varphi(z)\, u(z)\rangle_{\mathscr{L}(\varphi)} - \langle h(z), u(z)\rangle_{\mathscr{L}(\psi)} = 2\langle f(z), u(z)\rangle$$

holds for every square summable element $u(z)$ of $\mathscr{L}(\psi)$. By Theorem 3, the identity

$$\langle B(z)\, g(z), u(z)\rangle_B - \langle [1 + B(z)]\, f(z), u(z)\rangle_B = -\langle f(z), u(z)\rangle$$

holds for every element $u(z)$ of $\mathscr{H}(B)$. Since the identity

$$\begin{aligned}\langle B(z)\, g(z), [1 - B(z)]\, u(z)\rangle_B &- \langle [1 + B(z)]\, f(z), [1 - B(z)]\, u(z)\rangle_B \\ &= -\langle f(z), [1 - B(z)]\, u(z)\rangle\end{aligned}$$

holds for every square summable element $u(z)$ of $\mathscr{L}(\psi)$, since

$$\langle f(z), \varphi(z)\, u(z)\rangle_{\mathscr{L}(\varphi)} = \langle [1 + B(z)]\, f(z), [1 - B(z)]\, u(z)\rangle_B\,,$$

and since

$$\langle g(z), u(z)\rangle = \langle f(z), [1 + B(z)]\, u(z)\rangle,$$

the identity

$$\langle B(z)\, g(z), [1 - B(z)]\, u(z)\rangle_B - \langle h(z), u(z)\rangle_{\mathscr{L}(\psi)} = \langle g(z), u(z)\rangle$$

holds for every square summable element $u(z)$ of $\mathscr{L}(\psi)$. By Theorem 3, $W(\psi, \varphi)$ takes $g(z)$ into $-h(z)$.

Proof of Theorem 5. These results are best seen from a schematic formulation of the theory of minimal decompositions of an element of $\mathscr{C}(z)$ with respect to $\mathscr{H}(B)$. Let T be multiplication by $B(z)$ as a transformation in $\mathscr{C}(z)$ and let T^* be its adjoint in $\mathscr{C}(z)$. Then the transformation $1 - TT^*$ coincides with the adjoint of the inclusion of $\mathscr{H}(B)$ in $\mathscr{C}(z)$. The transformation $1 - T^*T$ coincides with the adjoint of the inclusion of $\mathscr{L}(\theta)$ in $\mathscr{C}(z)$.

Consider the range $\mathscr{M}(1 + T)$ of multiplication by $1 + B(z)$ as a Hilbert space with the unique norm which makes multiplication by $1 + B(z)$ an isometry of $\mathscr{C}(z)$ onto $\mathscr{M}(1 + T)$. By the proof of Theorem 1, the series $[1 + B(z)]\, [1 + \bar{B}(w)]\, c/(1 - z\bar{w})$ belongs to $\mathscr{M}(1 + T)$ for every vector c when $|w| < 1$ and the identity

$$\bar{c}f(w) = \langle f(z), [1 + B(z)]\, [1 + \bar{B}(w)]\, c/(1 - z\bar{w})\rangle_{\mathscr{M}(1+T)}$$

holds for every element $f(z)$ of the space.

Let $\mathscr{H}$ be the Hilbert space consisting of all elements $h(z)$ of $\mathscr{C}(z)$ of the form $h(z) = f(z) + g(z)$ with $f(z)$ in $\mathscr{H}(B)$ and $g(z)$ in $\mathscr{M}(1 + T)$,

$$\| h(z)\|_{\mathscr{H}}^2 = \inf\{\| f(z)\|_B^2 + \| g(z)\|_{\mathscr{M}(1+T)}^2\}$$

where the infimum is taken over all such representations $h(z) = f(z) + g(z)$. By the theory of minimal decompositions, the series

$$\begin{aligned}&[2 + B(z) + \bar{B}(w)]\, c/(1 - z\bar{w}) \\ &\quad = [1 - B(z)\, \bar{B}(w)]\, c/(1 - z\bar{w}) + [1 + B(z)]\, [1 + \bar{B}(w)]\, c/(1 - z\bar{w})\end{aligned}$$

belongs to $\mathscr{H}$ for every vector c when $|w| < 1$ and the identity

$$\bar{c}f(w) = \langle f(z), [2 + B(z) + \bar{B}(w)]\, c/(1 - z\bar{w})\rangle_{\mathscr{H}}$$

holds for every element $f(z)$ of $\mathscr{H}$. It follows that $\mathscr{H}$ is isometrically equal to $\mathscr{L}(2 + 2B)$.

This construction is used to compute the adjoint of the inclusion of $\mathscr{L}(2 + 2B)$ in $\mathscr{C}(z)$. Since the adjoint of the inclusion of $\mathscr{H}(B)$ in $\mathscr{C}(z)$ is $1 - TT^*$ and since the adjoint of the inclusion of $\mathscr{M}(1 + T)$ in $\mathscr{C}(z)$ is $(1 + T)(1 + T^*)$, the adjoint of the inclusion of $\mathscr{L}(2 + 2B)$ in $\mathscr{C}(z)$ is $2 + T + T^*$.

The same constructions are now made with T replaced by T^*. Let $\mathscr{M}(1 + T^*)$ be the range of $1 + T^*$ in the unique norm which makes $1 + T^*$ a partial isometry of $\mathscr{C}(z)$ onto $\mathscr{M}(1 + T^*)$. Then $\mathscr{L}(2 + 2B)$ is the set of elements $h(z)$ of $\mathscr{C}(z)$ of the form $h(z) = f(z) + g(z)$ with $f(z)$ in $\mathscr{L}(\theta)$ and $g(z)$ in $\mathscr{M}(1 + T^*)$,

$$\| h(z)\|_{\mathscr{L}(2+2B)}^2 = \inf\{\| f(z)\|_{\mathscr{L}(\theta)}^2 + \| g(z)\|_{\mathscr{M}(1+T^*)}^2\},$$

the infimum taken over all such representations $h(z) = f(z) + g(z)$.

It follows that the inclusion of $\mathscr{L}(\theta)$ in $\mathscr{L}(2 + 2B)$ does not increase norms. A space $\mathscr{L}(2 + 2B - \theta)$ exists and is isometrically to $\mathscr{M}(1 + T^*)$. The elements of $\mathscr{L}(\theta)$ which have the same norm in $\mathscr{L}(\theta)$ as in $\mathscr{L}(2 + 2B)$ are the orthogonal complement of $\mathscr{L}(2 + 2B - \theta)$ in $\mathscr{L}(2 + 2B)$. They are also the orthogonal complement in $\mathscr{L}(\theta)$ of the elements of $\mathscr{L}(\theta)$ in $\mathscr{L}(2 + 2B - \theta)$.

Since $1 + T^*$ takes $[f(z) - f(0)]/z$ into $[g(z) - g(0)]/z$ whenever it takes $f(z)$ into $g(z)$, it takes polynomials into polynomials without increasing degrees. Since polynomials are dense in the domain of $1 + T^*$, they are dense in its range. Since the operator $1 + B(0)$ has an everywhere defined and bounded inverse, all elements of $\mathscr{C}$ belong to the range of $1 + T^*$. It follows that all polynomials with vector coefficients belong to the range of $1 + T^*$.

Since $\mathscr{L}(\theta)$ is contained in $\mathscr{L}(2 + 2B)$ and since the inclusion does not increase norms, it follows from the definition of extension spaces that $\mathscr{E}(\theta)$ is contained in $\mathscr{E}(2 + 2B)$ and that the inclusion does not increase norms. For the same reason $\mathscr{E}(2+2B-\theta)$ is contained in $\mathscr{E}(2+2B)$ and the inclusion does not increase norms. Every element $(u(z), v(z))$ of $\mathscr{E}(2+2B)$ is of the form

$$(u(z), v(z)) = (f(z), g(z)) + (h(z), k(z))$$

with $(f(z), g(z))$ in $\mathscr{E}(\theta)$ and $(h(z), k(z))$ in $\mathscr{E}(2+2B-\theta)$. The inequality

$$\|(u(z), v(z))\|^2_{\mathscr{E}(2+2B)} \leqslant \|(f(z), g(z))\|^2_{\mathscr{E}(\theta)} + \|(h(z), k(z))\|^2_{\mathscr{E}(2+2B-\theta)}$$

is satisfied for every such decomposition. A minimal decomposition always exists for which equality holds.

The elements of $\mathscr{E}(\theta)$ which have the same norm in $\mathscr{E}(\theta)$ as in $\mathscr{E}(2+2B)$ are the orthogonal complement of $\mathscr{E}(2+2B-\theta)$ in $\mathscr{E}(2+2B)$. They are also the orthogonal complement in $\mathscr{E}(\theta)$ of the elements of $\mathscr{E}(\theta)$ which belong to $\mathscr{E}(2+2B-\theta)$.

Since all polynomials belong to $\mathscr{L}(2+2B-\theta)$, every element $(f(z), g(z))$ $\mathscr{E}(2+2B)$ such that $f(z)$ is a polynomial belongs to the closure of $\mathscr{E}(2+2B-\theta)$ in $\mathscr{E}(2+2B)$. Every element $(f(z), g(z))$ of $\mathscr{E}(\theta)$ such that $f(z)$ is a polynomial belongs to the closure in $\mathscr{E}(\theta)$ of the elements of $\mathscr{E}(\theta)$ which belong to $\mathscr{E}(2+2B-\theta)$.

It follows that if an element $(f(z), g(z))$ of $\mathscr{E}(\theta)$ has the same norm in $\mathscr{E}(\theta)$ as in $\mathscr{E}(2+2B)$, then $f(z)$ has the same norm in $\mathscr{L}(\theta)$ as $(f(z), g(z))$ does in $\mathscr{E}(\theta)$, $f(z)$ has the same norm in $\mathscr{L}(2+2B)$ as $(f(z), g(z))$ does in $\mathscr{E}(2+2B)$, and $f(z)$ has the same norm in $\mathscr{L}(\theta)$ as in $\mathscr{L}(2+2B)$.

If $f(z)$ is an element of $\mathscr{L}(\theta)$ which has the same norm in $\mathscr{L}(\theta)$ as in $\mathscr{L}(2+2B)$, then an element $g(z)$ of $\mathscr{C}(z)$ exists such that $(f(z), g(z))$ belongs to $\mathscr{E}(\theta)$ and has the same norm in $\mathscr{E}(\theta)$ as $f(z)$ does in $\mathscr{L}(\theta)$. Since the norm of $(f(z), g(z))$ in $\mathscr{E}(2+2B)$ cannot be less than the norm of $f(z)$ in $\mathscr{L}(2+2B)$, the norm of $(f(z), g(z))$ in $\mathscr{E}(\theta)$ is equal to the norm of $(f(z), g(z))$ in $\mathscr{E}(2+2B)$.

Proof of Theorem 6. If $P(e^{it})$ is a measurable operator valued function of real t, which is periodic of period 2π, and if its values are nonnegative operators which are bounded independently of t, let $L^2(P)$ be the Hilbert space of (equivalence classes of) measurable vector valued functions $f(e^{it})$ of real t, which are periodic of period 2π, such that

$$2\pi \|f\|_P^2 = \int_0^{2\pi} \|f(e^{it})\|^2_{\mathscr{C}(P(e^{it}))}\, dt < \infty.$$

When $P(e^{it}) = 1$ identically, the usual Hilbert space $L^2(1)$ of square integrable vector valued functions is obtained. In the general case, $L^2(P)$ is contained in $L^2(1)$. Completeness of the space follows from the completeness of $L^2(1)$.

Note that when $f(e^{it}) = P(e^{it})\, g(e^{it})$ for a measurable vector valued function $g(e^{it})$,

$$2\pi \| f \|_P^2 = \int_0^{2\pi} \bar{g}(e^{it})\, P(e^{it})\, g(e^{it})\, dt.$$

The space $L^2(P)$ is applied in connection with a space $\mathscr{L}(\varphi)$ when the Poisson representation of $\operatorname{Re} \varphi(w)$ is

$$\operatorname{Re} \varphi(w) = (1 - | w |^2)/2\pi \int_0^{2\pi} P(e^{it})\, dt/| 1 - e^{-it} w |^2,$$

for $| w | < 1$. Since every function $f(e^{it})$ which belongs to $L^2(P)$ belongs to $L^2(1)$, it is the boundary value function of an element $(f(z), g(z))$ of $\mathscr{E}(1)$. This element belongs to $\mathscr{E}(\varphi)$ and the transformation taking $f(e^{it})$ into $(f(z), g(z))$ so defined is an isometry of $L^2(P)$ onto $\mathscr{E}(\varphi)$. The verification is made in an obvious way using particular elements of $L^2(P)$ which yield computable integrals when used for inner products. These are for every vector c and every number w, $| w | < 1$, $P(e^{it})\, c/(1 - e^{it}\bar{w})$, which is the boundary value function of

$$(\tfrac{1}{2}[\varphi(z) + \bar{\varphi}(w)]\, c/(1 - z\bar{w}), \tfrac{1}{2}[\varphi^*(z) - \varphi(\bar{w})]\, c/(z - \bar{w})),$$

and $P(e^{it})\, c/(e^{it} - \bar{w})$, which is the boundary value function of

$$(\tfrac{1}{2}[\varphi(z) - \varphi(\bar{w})]\, c/(z - \bar{w}), \tfrac{1}{2}[\varphi^*(z) + \varphi(\bar{w})]\, c/(1 - z\bar{w})).$$

The boundary value function theory is applied first in the case $\varphi(z) = 2 + 2B(z)$ and $P(e^{it}) = 2 + B(e^{it}) + \bar{B}(e^{it})$. If states that an element $(f(z), g(z))$ of $\mathscr{E}(1)$ belongs to $\mathscr{E}(2 + 2B)$ if, and only if, its boundary value function $f(e^{it})$ satisfies the condition

$$\int_0^{2\pi} \| f(e^{it}) \|^2_{\mathscr{C}(2+B(e^{it})+\bar{B}(e^{it}))}\, dt < \infty,$$

in which case the integral is equal to

$$2\pi \| (f(z), g(z)) \|^2_{\mathscr{E}(2+2B)} \, .$$

The boundary value function theory is next applied when $P(e^{it}) = [1 + \bar{B}(e^{it})]\, [1 + B(e^{it})]$ in order to compute the corresponding function $\varphi(z)$. In this case an element $(f(z), g(z))$ of $\mathscr{E}(1)$ belongs to $\mathscr{E}(\varphi)$ if, and only if, its boundary value function $f(e^{it})$ is of the form $f(e^{it}) = [1 + \bar{B}(e^{it})]\, h(e^{it})$ for an element $h(e^{it})$ of $L^2(1)$. The norm of $(f(z), g(z))$ in $\mathscr{E}(\varphi)$ is then equal to the norm of $h(e^{it})$ in $L^2(1)$. By the proof of Theorem 5, $\varphi(z) = 1 + B(z) - \theta(z)$.

The boundary value function theory is also applied in the case that $P(e^{it}) = 1 - \bar{B}(e^{it})\, B(e^{it})$, in which case $\varphi(z) = \theta(z)$ by the proof of Theorem 5. An element $(f(z), g(z))$ of $\mathscr{E}(1)$ belongs to $\mathscr{E}(\theta)$ if, and only if, its boundary value function $f(e^{it})$ satisfies the condition

$$\int_0^{2\pi} \|f(e^{it})\|^2_{\mathscr{C}(1-\bar{B}(e^{it})B(e^{it}))}\, dt < \infty,$$

in which case the integral is equal to

$$2\pi \|(f(z), g(z))\|^2_{\mathscr{E}(\theta)} .$$

Proof of Theorem 7. The theorem restates, in a different notation, results which have already been verified in the proof of Theorem 5.

Proof of Theorem 8. The theorem is proved by a schematic use of the theory of minimal decompositions, as in the proof of Theorem 5. Let T be multiplication by $B(z)$ as a transformation in $\mathscr{C}(z)$ and let T^* be its adjoint in $\mathscr{C}(z)$. Then

$$(1 - \lambda^2)(1 + \lambda T^*)^{-1}(1 - T^*T)(1 + \lambda T)^{-1}$$

is the adjoint of the inclusion of $\mathscr{L}(\theta_\lambda)$ in $\mathscr{C}(z)$. By Theorems 1, 3, and 4, $1 + T^*$ takes every square summable element $f(z)$ of $\mathscr{L}(1 + \varphi)$ into an element $g(z)$ of $\mathscr{L}(2 + 2B)$ which has the same norm in $\mathscr{L}(2 + 2B)$ as $f(z)$ does in $\mathscr{L}(1 + \varphi)$. Since the elements of $\mathscr{L}(\theta_\lambda)$ are square summable and since $2 + T + T^*$ is the adjoint of the inclusion of $\mathscr{L}(2 + 2B)$ in $\mathscr{C}(z)$, the conclusion of the theorem is equivalent to the inequality

$$(1 + T^*)(1 + \lambda T^*)^{-1}(1 - T^*T)(1 + \lambda T)^{-1}(1 + T) \leqslant 2 + T + T^*,$$

which follows from the hypotheses of the theorem because they can be written

$$\begin{aligned}(1 - \lambda)^2(1 - T^*T) + (1 - \lambda)^2(1 + T^*)\, T^*T(1 + T)& \\ + \lambda(1 - \lambda)(1 - 2T^*T + T^{*2}T^2) + \lambda(1 + T^*)^2(1 + T)^2& \\ \geqslant 0.&\end{aligned}$$

Proof of Theorem 9. The schematic notation from the proofs of Theorems 5 and 8 is again used. Let T be multiplication by $B(z)$ as a transformation in $\mathscr{C}(z)$ and let T^* be its adjoint in $\mathscr{C}(z)$. By Theorem 5, it must be shown that an element f of $\mathscr{L}(\theta)$ vanishes if it has the same norm in $\mathscr{L}(\theta)$ as in $\mathscr{L}(2 + 2B)$, or equivalently if it is orthogonal in $\mathscr{L}(2 + 2B)$ to the range of $1 + T^*$.

By Theorem 2, $(1 + \lambda T^*)^{-1} f$ belongs to $\mathscr{L}(\theta_\lambda)$ and

$$(1 - \lambda^2)\|(1 + \lambda T^*)^{-1} f\|^2_{\mathscr{L}(1+\varphi)} \leqslant \|f\|^2_{\mathscr{L}(\theta_\lambda)} .$$

By Theorem 8,

$$\|(1 + \lambda T^*)^{-1} f\|_{\mathscr{L}(1+\varphi)} \leqslant \|f\|_{\mathscr{L}(\theta)} .$$

As in the proof of Theorem 2, $(1 + T^*)(1 + \lambda T^*)^{-1} f$ belongs to $\mathscr{L}(2 + 2B)$ and

$$\|(1 + T^*)(1 + \lambda T^*)^{-1} f\|_{\mathscr{L}(2+2B)} \leqslant \|f\|_{\mathscr{L}(2+2B)}.$$

Since as λ approaches one,

$$\lim \langle (1 + T^*)(1 + \lambda T^*)^{-1} f, g\rangle_{\mathscr{L}(2+2B)} = \langle f, g\rangle_{\mathscr{L}(2+2B)}$$

for a dense set of elements g of $\mathscr{L}(2 + 2B)$, the same conclusion holds for every element g of $\mathscr{L}(2 + 2B)$ and

$$\lim\, (1 + T^*)(1 + \lambda T^*)^{-1} f = f$$

in the metric of $\mathscr{L}(2 + 2B)$. Since f is orthogonal in $\mathscr{L}(2 + 2B)$ to every one of the approximating elements by construction, it is zero.

Proof of Theorem 10. Since

$$[\varphi(z) + \bar{\varphi}(w)]/(1 - z\bar{w}) = (1 - zU)^{-1} [\varphi(z) + \bar{\varphi}(0)] (1 - \bar{w}\bar{U})^{-1},$$

the hypotheses imply that the real part of $\varphi(w)$ is nonnegative in the unit disk. So a space $\mathscr{L}(\varphi)$ exists. The identity $[f(z) - f(0)]/z = Uf(z)$ is easily verified when $f(z) = [\varphi(z) + \bar{\varphi}(w)]\, c/(1 - z\bar{w})$ for a vector c and a number w, $|w| < 1$. The identity follows by linearity and continuity for all elements $f(z)$ of $\mathscr{L}(\varphi)$.

Proof of Theorem 11. Since

$$[1 - B(z)\bar{B}(w)]/(1 - z\bar{w}) = [1 - zS\bar{B}(0)]^{-1} [1 - B(0)\bar{B}(0)] [1 - \bar{w}B(0)\bar{S}]^{-1},$$

the operator $B(w)$ is bounded by one in the unit disk. So a space $\mathscr{H}(B)$ exists. The identity $[f(z) - f(0)]/z = S\bar{B}(0) f(0)$ is easily verified when $f(z) = [1 - B(z)\bar{B}(w)]\, c/(1 - z\bar{w})$ for a vector c and a number w, $|w| < 1$. The identity follows by linearity and continuity for all elements $f(z)$ of $\mathscr{H}(B)$.

The remainder of the proof depends on the identity

$$[S - zB(0)] [\bar{B}(0) - z\bar{S}] = [S\bar{B}(0) - z] [1 - zB(0)\bar{S}]$$

which follows from the hypotheses of the theorem. It follows that

$$[S - zB(0)] [1 - B^*(z) B(\bar{w})] [\bar{S} - \bar{w}\bar{B}(0)]/(1 - z\bar{w}) = 1 - B(0)\bar{B}(0).$$

The identity $[S - zB(0)] f(z) = Sf(0)$ is easily verified for an element of $\mathscr{H}(B^*)$ of the form

$$[1 - B^*(z) B(\bar{w})] [\bar{S} - \bar{w}\bar{B}(0)]\, c/(1 - z\bar{w})$$

for a vector c and a number w, $|w| < 1$. The identity follows by linearity and continuity in the closed span of these elements. But if an element $f(z)$ of $\mathscr{H}(B^*)$ is orthogonal to these elements, $[S - zB(0)]f(z) = 0$, and the identity holds in this case also. So the identity holds for every element $g(z)$ of $\mathscr{H}(B^*)$.

Since $\mathscr{L}(\theta^*)$ is contained in $\mathscr{H}(B^*)$, the same identity holds for every element $f(z)$ of $\mathscr{L}(\theta^*)$. By the theory of extension spaces, the identity $[B(0) - Sz]f(z) = B(0)f(0)$ holds for every element $f(z)$ of $\mathscr{L}(\theta)$.

Proof of Theorem 12. These results are verified by the obvious calculations.

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